An Enhancement in Sum-of-Squares Optimization based Region of Attraction Estimation for Power Systems

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***Abstract*—To measure transient stability of power systems, estimation of its region of attraction (RoA) is one of the common approaches used in practice. Among the class of available methods of RoA estimation, Sum-of-Squares (SoS) optimization based method has been introduced recently, and is promising due to its ability to compute an estimate of the RoA in a less restrictive and scalable way. In this paper, we propose an enhanced version of SoS optimization based RoA estimation. Our result shows that the proposed algorithm can significantly improve the size of the estimated RoA (more than 300%), and reduce the overall computation time (more than 60%), as compared to the state-of-the-art.**

***Index Terms*— Lyapunov function, power systems, region of attraction, Sum-of-Squares optimization, transient stability.**

I. Introduction

QUANTIFICATION of transient stability of power systems, is a key to understand its resilience, and has become more important in recent times, as both complexity and vulnerability of power grid is increasing due to rapid growth of distributed generation. In this paper, we measure transient stability of a power system operating at a stable equilibrium point (SEP), by the volume of the corresponding RoA, namely the set of states from where the power system can return to the SEP, if perturbed. The problem of estimating RoA of power systems has been studied intensively over the last few decades, and can be broadly classified into two categories, 'non-Lyapunov' and 'Lyapunov'. Among the 'non-Lyapunov' methods, [1] rely on a well-defined analytical energy function. This however, does not exist for system with transfer conductance. Also it requires the estimation of a critical energy, the value of which is difficult to compute reliably. The closest unstable equilibrium point (UEP) method [2] is contingent upon identification of the set of all unstable equilibrium points located on the boundary of RoA, which may be intractable for complex systems, and the estimated RoA is conservative. The controlling UEP method, supported by a subsequent advancement called boundary of stability region based controlling UEP (BCU) method [3], also turns out to be unreliable in several cases [4], [5], as it assumes linearity of the fault-on trajectory of the power system. On the other hand, among the 'Lyapunov' methods [6], Zubov's method solves a set of PDE to compute the RoA boundary. Even in this setting transfer conductance poses a challenge, when applied to power systems. Popov's stability criterion based method relies on satisfaction of sector conditions, where again the presence of transfer conductance presents impediment [7], [8]. [9] proposed a method of estimating RoA by propagation of boundary of a reachable set around the equilibrium, backward in time, by solving a HJI PDE using level set method, but the method relies on defining an analytical energy function that comes with the aforesaid limitation.

Thus aforementioned methods of computing RoA invariably, have complexities scaling exponentially with the dimension of the state space. Among the recent 'Lyapunov' methods, SoS optimization based method solves a set of SoS maximizations to estimate RoA iteratively, in form of an optimal sublevel set of a Lyapunov function that need not be known a priori. Unlike the energy function based methods, such algorithmic construction of RoA does not require the system model to be free of transfer conductance. Also the complexity of computing RoA using SoS optimization, is polynomial with respect to the dimension of the state space. This approach was introduced to power systems in [10], and a more compact implementation, was recently presented in [11].

In this paper, we propose an enhancement of the algorithm presented in [11], where we iteratively alternate over two SoS maximizations, which (i) enlarges the size of the estimated RoA, and (ii) reduces the overall computation time, both by significant amounts (as shown in TABLE II). We also introduce a superior stopping criterion, for estimating the RoA as an inscribing ellipsoid, which alleviates one of the assumptions in algorithm in [11]. The proposed algorithm is implemented on the modified WSCC-9 bus test system to validate the enhancements with respect to the state-of-the-art.

II. Preliminaries

A. Model

In dynamic analysis of a power system comprising of N generators, it is common to study the system with reduced $n = N-1$ number of generators, while setting the remaining one as the fixed inertial frame spinning at the synchronous speed. In this setting, we model the dynamics of the system by the classical swing equations, which for the i^{th} generator, where $i \in \{1,\dots,n\}$, can be written as:

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$$M_i\ddot{\delta}_i + D_i\dot{\delta}_i + P_{Gi} = P_{mi}, \quad (1)$$

where δ_i denotes the angle of the generator emf phasor $E_i\angle\delta_i$ (behind its transient reactance), measured relative to the inertial frame, and M_i, D_i, and P_{Mi} are constants denoting inertia, damping and mechanical power input respectively, while P_{Gi} is the electrical power output given by:

$$P_{Gi} = \sum_{j=1}^{N} E_i E_j Y_{ij}\cos(\theta_{ij} - \delta_i + \delta_j), \quad (2)$$

where for a pair of generators $(i,\ j)$, $Y_{ij}\angle\theta_{ij}$ denotes the $(i,j)^{th}$ element of the Kron-reduced admittance matrix. E_i is kept constant at its value corresponding to the equilibrium point of (1) and (2) (also seen by setting $\ddot{\delta}_i = \dot{\delta}_i = 0$), found by solving the power flow (PF) equations, and the value of P_{Mi} is also held constant, at the value of P_{Gi} at the equilibrium.

B. Notations

$\mathbb{R}$ (resp., $\mathbb{R}_+$ and $\mathbb{R}_{++}$) denotes the space of reals (resp., non-negative reals and positive reals). $\mathbb{R}^n$ denotes the real field of dimension *n*. $\mathbb{P}_x$ is the set of polynomials with coefficients over the field of reals defined over a vector valued indeterminate *x*, and $\mathbb{P}_{x,0} \subset \mathbb{P}_x$ is the subset of those polynomials that evaluate to zero at the origin. $\mathbb{P}_x^n$ denotes *n* dimensional space of polynomials, where each element lies in $\mathbb{P}_x$. Our RoA computation relies on a certain class of polynomials, called Sum-of-squares (SoS) as defined below:

Definition 1*: For a non-negative integer* d *and* $x \in \mathbb{R}^{2n}$, *a homogeneous polynomial* $s(x)$ *of degree* $2d$, *is called* ***Sum-of-Squares polynomial****, if and only if there exist polynomials* $u_1(x),...,u_r(x)$, *each of degree* d, *such that* $s(x) = \sum_{i=1}^{r} u_i(x)^2$.

We let $\mathbb{S}_x \subset \mathbb{P}_x$ to denote the set of all SoS polynomials in $\mathbb{P}_x$.

III. Problem Formulation

A. Background

For a power system with dynamics defined by (1) and (2), the state vector consists of the generator angles and speeds, and is denoted as $x = [\delta_1\ \dot{\delta}_1\ \delta_2\ \dot{\delta}_2\ ...\ \delta_n\ \dot{\delta}_n]^T$, and its nonlinear dynamics may be viewed as:

$$\dot{x} = f(x(t)), \quad (3)$$

where $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a locally Lipschitz nonlinear map, typically defined over a domain $D \subset R^{2N}$. $x_0 \in \mathbb{R}^{2N}$ is an equilibrium point if there is no rate of change at that state, i.e. if $f(x_0) = 0$.

Definition 2 [12]*: For an autonomous system defined by* (3), *the* ***region of attraction (RoA)*** *corresponding to its SEP* x_0, *is the largest subset* Ω *of the domain* D, *such that if the system starts at any state in* Ω, *it eventually reaches* x_0 *and remains stable.*

The true RoA Ω corresponding to a SEP x_0 is an open set, boundary of which is defined by the union of stable manifolds of the unstable equilibrium points around the SEP. If an autonomous system is initialized at a point outside Ω, then it never eventually enters Ω, and thus can never reach x_0. Accordingly, if a stable power system is perturbed due to an occurrence of a fault (resp., any other transient disturbance), the operating point after clearance of the fault (resp., withdrawal of the cause of disturbance), must lie within the RoA corresponding to the post-clearance (resp., post-withdrawal) SEP, for the post-clearance (resp., post-withdrawal) system to eventually reach the SEP. Hence transient stability margin of a power system, against the occurrence of a fault (resp., any other disturbance), can be measured by estimating the volume of the RoA corresponding to the SEP of the post-clearance (resp., post-withdrawal) system. It is common to characterize the RoA of (3) by a Lyapunov function as defined below:

Definition 3 [12]: $V(x)$, *a continuously differentiable real valued scalar function of state vector* $x \in D$, *is a* ***Lyapunov function*** *corresponding to a SEP* x_0 *of an autonomous system defined by* (3), *if the following conditions* (4)-(7) *are satisfied*:

$$V(x_0) = 0, \quad (4)$$

$$V(x) > 0 \qquad \forall x \in D \backslash \{x_0\}, \quad (5)$$

$$\dot{V}(x) \le 0 \qquad \forall x \in D, \quad (6)$$

$$\dot{V}(x_0) = 0, \quad (7)$$

and x_0 *is called asymptotically stable if the following stronger version of* (6) *holds*:

$$\dot{V}(x) < 0 \qquad \forall x \in D \backslash \{x_0\}. \quad (8)$$

Using a Lyapunov function, the RoA can be characterized by:

$$RoA := \{x \in D \mid (4),(5),(7),(8)\}. \quad (9)$$

Next we state the theorem that is instrumental for all SoS optimization based RoA estimation approaches in practice:

Theorem 1 (Positivstellensatz) [13]: *Let the* i^{th} *polynomial in a set of* p *(resp.,* q*) polynomials in* $\mathbb{P}_x$, *be denoted by* $g_i(x)$ *(resp.,* $h_i(x)$*). Let* $\mathcal{S}_i$ *be the* i^{th} *subset of the set of all subsets of the integers* $\{1,...,n\}$, *excluding the null set. Then* $\{x \in \mathbb{R}^{2n} \mid g_i(x) \ge 0\ \forall 1 \le i \le p,\ h_i(x) = 0\ \forall 1 \le i \le q\}$ *defines a semi-algebraic set, which is empty if and only if there exist* $v_i(x) \in \mathbb{P}_x\ \forall 1 \le i \le p$, *and* $s_i(x) \in \mathbb{S}_x\ \forall 0 \le i \le 2^q - 1$ *satisfying the following*:

$$\sum_{i=1}^{p} v_i(x)h_i(x) + \sum_{i=1}^{2^q-1} s_i(x)\prod_{j\in\mathcal{S}_i} g_j(x) + s_0(x) = -1. \quad (10)$$

The complexity of certifying emptiness of the above semi-algebraic set, by finding the polynomials $v_i(x)$ and $s_i(x)$ that satisfy (10), is exponential with respect to the degree of polynomials $v_i(x)$ and $s_i(x)$, in general. However, a simple algebraic operation transforms (10) into a SoS constraint as follows:

$$-\sum_{i=1}^{m} v_i(x)h_i(x) - \sum_{i=1}^{2^n-1} s_i(x)\prod_{j\in\mathcal{S}_i} g_j(x) \in \mathbb{S}_x, \quad (11)$$

which can be expressed as a set of linear matrix inequality (LMI) [14], by affine parameterization of polynomials $v_i(x)$

and $s_i(x)$ [13]. By restricting the degree of the polynomials $v_i(x)$ and $s_i(x)$, the above emptiness checking problem can be reduced into a semidefinite program (SDP) [14] that can be solved efficiently. [13] suggests a systematic hierarchy of polynomial sized SDP relaxations, to solve emptiness checking of the general class of semi-algebraic sets defined by finite set of constraints of the form (11).

B. Inner Approximation of the RoA

Numerically computing RoA is in general of exponential complexity, in the dimension of the state space. Finding an inner approximation of the RoA can be formulated as the following optimization problem [11], where without loss of generality, its SEP can be taken to be the origin (otherwise one can simply employ a linear shift of coordinates):

$$\max_{V\in\mathbb{P}_{x,0},\, c\in\mathbb{R}_{++}} \gamma$$

subject to

$$\{x\in\mathbb{R}^{2n} \mid V(x)\le 0, x\ne 0\}=\varnothing \tag{12}$$
$$\{x\in\mathbb{R}^{2n} \mid p_+(x)\le\gamma, V(x)\ge c, V(x)\ne c\}=\varnothing$$
$$\{x\in\mathbb{R}^{2n} \mid V(x)\le c, \dot{V}(x)\ge 0, x\ne 0\}=\varnothing,$$

where $p_+(x)$ is a given positive semidefinite polynomial over x, such that it is positive everywhere except at the origin, and $\gamma\in\mathbb{R}_{++}$. In (12), the first constraint corresponds to conditions (4)-(5), the third constraint corresponds to conditions (5)-(7), and the second constraint implies that the set $\{x \mid p_+(x)\le\gamma\}$ is contained within the set $\{x \mid V(x)\le c\}$. Note that:

$$\dot{V}(x)=\frac{\partial V}{\partial x}\dot{x}=\frac{\partial V}{\partial x}f(x), \tag{13}$$

which implies that the system dynamics is implicitly captured in the third constraint of (12).

In the case of power systems, the set defined by the second constraint of (12) is not semi-algebraic, since $\dot{V}(x)$ is non-polynomial (includes trigonometric functions). Hence *Theorem 1* cannot be directly applied to cast all the constraints in (12) as SoS constraints of the form (11). However, the following nonlinear transformation, as suggested in [10], [11], for all $i\in\{1,\dots,n\}$, "polynomializes" the dynamic equations (3):

$$\begin{aligned} z_{3i-2} &= \sin(\delta_i),\\ z_{3i-1} &= \dot{\delta}_i,\\ z_{3i} &= 1-\cos(\delta_i). \end{aligned} \tag{14}$$

After the transformation, the power system dynamics is:

$$\dot{z}(t)=\bar{f}(z(t)), \tag{15}$$
$$g(z(t))=0, \tag{16}$$

where $z\in\mathbb{R}^{3n}$, and $\bar{f}:\mathbb{R}^{3n}\to\mathbb{R}^{3n}$, $g:\mathbb{R}^{3n}\to\mathbb{R}^{n}$ are vectors of polynomial maps with elements satisfying the following for $i\in\{1,\dots,n\}$:

$$\begin{aligned} \bar{f}_{3i-2}(z) &= (1-z_{3i})z_{3i-1},\\ \bar{f}_{3i-1}(z) &= (P_{mi}-P_{Gi}-D_i z_{3i-1})/M_i,\\ \bar{f}_{3i}(z) &= z_{3i-2}z_{3i-1}, \end{aligned} \tag{17}$$

and

$$g_i(z)=z_{3i-2}^2+z_{3i}^2-2z_{3i}, \tag{18}$$

In the above equations, for all $i\in\{1,\dots,n\}$, P_{Gi} is given by:

$$\begin{aligned} P_{Gi} = \sum_{j=1}^{N} E_i E_j \big(& B_{ij}(z_{3i-2}(1-z_{3j})-z_{3j-2}(1-z_{3i}))+\\ & C_{ij}((1-z_{3i})(1-z_{3j})-z_{3i-2}z_{3j-2}\big), \end{aligned} \tag{19}$$

where $B_{ij}=Y_{ij}\sin\theta_{ij}$, and $C_{ij}=Y_{ij}\cos\theta_{ij}$. Accordingly, (12) can be rephrased for the transformed space as follows:

$$\max_{V\in\mathbb{P}_{z,0},\, c\in\mathbb{R}_{++}} \gamma$$

subject to

$$\{z\in\mathbb{R}^{3n} \mid V(z)\le 0,\ g(z)=0,\ z\ne 0\}=\varnothing \tag{20}$$
$$\{z\in\mathbb{R}^{3n} \mid p_+(z)\le\gamma,\ V(z)\ge c,\ g(z)=0,\ V(z)\ne c\}=\varnothing$$
$$\{z\in\mathbb{R}^{3n} \mid V(z)\le c,\ \dot{V}(z)\ge 0,\ g(z)=0,\ z\ne 0\}=\varnothing.$$

Now that all the constraints in (20) are semi-algebraic, *Theorem 1* can be applied (see the corollary to *Theorem 1* in (11)) to obtain the following constrained version:

$$\max_{V\in\mathbb{P}_{z,0},\ v_1,v_2,v_3\in\mathbb{P}_z^N,\ s_1,s_2,s_3\in\mathbb{S}_z,\ c\in\mathbb{R}_{++}} \gamma \tag{21}$$

subject to

$$V(z)-v_1^T(z)g(z)-q_+(z)\in\mathbb{S}_z \tag{22}$$
$$-s_1(z)(\gamma-p_+(z))-v_2^T(z)g(z)-(V(z)-c)\in\mathbb{S}_z \tag{23}$$
$$-s_2(z)(c-V(z))-s_3(z)\dot{V}(z)-v_3^T(z)g(z)-q_+(z)\in\mathbb{S}_z, \tag{24}$$

where $q_+(z)$ is a polynomial with properties same as $p_+(x)$. Due to the presence of the product of polynomial variables in the first two terms of the last constraint, (21)-(24) is non-convex in general. An iterative algorithm to find a local optimal solution of (21) is provided in [11], with a proof of convergence, and the reader is referred to [11] for the steps in the algorithm in [11]. Next, we present our own version of the algorithm that enhances the one given in [11] by improving the inner approximation of the volume of RoA, and also reducing the computation time, significantly. We first provide our algorithm and next discuss the resulting enhancements.

***Our algorithm*:**

(i) Select $p_+(x)$, $q_+(z)$, scaling factor $0<\eta<1$, and small enough ϵ, $\epsilon_{\mathcal{V}}$, $\alpha_{max}\in\mathbb{R}_{++}$ $(\alpha_{max}>\epsilon)$. Set $c_0=0$, $\gamma_0=0$, $u=1$, $v=1$, $w=1$, $\alpha=\alpha_{max}$. Select a polynomial $r_+(z)$, with properties same as $p_+(x)$.

(ii) Find: $V\in\mathbb{P}_{z,0}$, $v_1,v_2,v_3\in\mathbb{P}_z^n$, $s_2\in\mathbb{S}_z$, such that: (25)

$$V(z)-v_1^T(z)g(z)-q_+(z)\in\mathbb{S}_z$$
$$-s_2(z)(\beta-r_+(z))-\dot{V}(z)-v_3^T(z)g(z)-q_+(z)\in\mathbb{S}_z.$$

(iii) (a) Repeat until ((26) is feasible or $\alpha<\epsilon$):

- Set $c_u=c_{u-1}+\alpha$, store c, $s_2(z)$, $s_3(z)$
- Find: $v_2,v_3\in\mathbb{P}_z^n$, $s_1,s_2,s_3\in\mathbb{S}_z$, (26)
  such that (23)-(24) holds.
- Set $u=u+1$, $\alpha=\alpha\times\eta$

(b) Set $\alpha=\alpha_{max}$, $c=c_{u-1}$, if $\alpha<\epsilon$, retrieve c, $s_2(z)$, $s_3(z)$

(iv) (a) Repeat until ((27) is feasible or $\alpha<\epsilon$):

- Set $\gamma_u = \gamma_{u-1} + \alpha$, store $\gamma, V(z)$
- Find : $V \in \mathbb{P}_{z,0}$, $v_1, v_2, v_3 \in \mathbb{P}_z^N$, $s_1 \in \mathbb{S}_z$, (27)
  such that (22)-(24) holds.
- Set $v = v+1$, $\alpha = \alpha \times \eta$

(b) Set $\alpha = \alpha_{max}, \gamma = \gamma_{u-1}$, if $\alpha < \epsilon$, retrieve $\gamma, V(z)$

(v) Compute $\mathcal{V}^w$ = volume of set $\{z \mid p_+(z) \le \gamma_{v-1}\}$ using (31). If $w > 1$ and $\mathcal{V}^w - \mathcal{V}^{w-1} < \epsilon_{\mathcal{V}}$, report $V(z) \le c$. Otherwise, set $p_+(z) = V(z)$, $\gamma = c, \gamma_{v-1} = \gamma$, $w = w+1$ and go to step (iii). ∎

The two maximizations SoSP1 and SoSP4 of the algorithm in [11], can be seen as "exploitation" and "exploration" respectively. SoSP1 exploits a given Lyapunov candidate to enlarge the estimate of RoA, and converges to its largest sublevel set contained within the true RoA. SoSP4 explores the polynomial space for a better Lyapunov candidate, and eventually converges to the maximal candidate that inscribes the true RoA, while containing the region $\{z \mid p_+(z) \le \gamma\}$. We replace SoSP1 and SoSP4 of the algorithm in [11] by the steps: (iii) and (iv) of our algorithm respectively, where we perform one step improvements of the respective objectives, instead of maximizing those. In light of the non-convexity of (21)-(24), the proposed modification enables a better balance between the exploitation and exploration, and results in a superior estimation of the RoA, and a faster convergence rate. The algorithm in [11] stops when the reduction in the value of the maximum coefficient of the polynomial $p_+(z)$, between the consecutive iterations, is below certain threshold. Here [11] implicitly assumes that the maximum coefficient decreases monotonically in every iteration of the outer loop, while no such formal proof exists. We modify the stopping criterion, by setting the algorithm to stop when the increase in the volume of estimated RoA between the consecutive iterations, is below $\epsilon_{\mathcal{V}}$. Since the volume is guaranteed to increase monotonically in every iteration [11], the modified stopping criterion relaxes the said implicit assumption of the algorithm in [11]. We exclude the proof of convergence, due to limited space. However, it is straightforward to extend the convergence proof provided in [11], to proof the convergence of our algorithm.

C. Computation of Volume of the Estimated RoA:

If we restrict $V(z)$ to be quadratic in (25) and (27), then the output of our algorithm $V(z) \le c$, defines an ellipsoid $\mathcal{E}$ in the space of $z \in \mathbb{R}^{3n}$, expressed as follows:

$$\mathcal{E} = \{z \in \mathbb{R}^{3n} \mid z^T A z + 2b^T z \le c\}, \tag{28}$$

where $A \in \mathbb{R}^{3n \times 3n}$ is symmetric positive semidefinite, $b \in \mathbb{R}^{3n}$, and both are obtained from the coefficients of the already computed terms of $V(z)$. We formulate the problem of computing the volume of ellipsoid defined by (28), by parametrizing the ellipsoid as the image of unit Euclidean ball under an affine transformation as follows:

$$\mathcal{E} = \{Bz + d \mid \|z\|_2 \le 1\}, \tag{29}$$

where $B \in \mathbb{R}^{3n \times 3n}$ is symmetric positive semidefinite and $d \in \mathbb{R}^{3n}$. Then the volume of $\mathcal{E}$, denoted by $\mathcal{V}$, is given by:

$$\mathcal{V} = \det(B) \times \mathcal{V}_u, \tag{30}$$

where $\mathcal{V}_u$ denotes volume of the unit Euclidean ball in $\mathbb{R}^{3n}$, and $\det(B)$ denotes the determinant of B. The transformation of $\mathcal{E}$ from (28) to (29), can be viewed as a special case of the computation of a maximal volume ellipsoid inscribed in the intersection of multiple ellipsoids, addressed in [14]. Accordingly, we can obtain (29) by solving the following convex problem with LMI constraint, where $\lambda \in \mathbb{R}$ is an auxiliary variable:

$$\begin{aligned} &\min_{B,\,d,\,\lambda} \log\det(B^{-1}) \\ &\text{subject to} \\ &\begin{pmatrix} -\lambda + c + b^T A^{-1} b & 0 & (d + A^{-1}b)^T \\ 0 & \lambda & B \\ d + A^{-1}b & B & A^{-1} \end{pmatrix} \succeq 0. \end{aligned} \tag{31}$$

While one can compute the exact volume of $\mathcal{E}$, using the existing closed form formula for $\mathcal{V}_u$, it suffices to measure $\mathcal{V}$ by $\det(B)$ for comparison purpose, since $\mathcal{V}_u$ is just a constant.

IV. Results

We implement the proposed algorithm for RoA estimation for the WSCC 3-generator 9-bus system, modified by assuming the generator models to be of second order with uniform damping, and the ratio D_i / M_i for each generator is held at 0.1. The SEP is obtained by PF solution of the network, considering generator 1 as the slack generator. The Krone reduced variables obtained from the PF solutions, are shown in TABLE I.

TABLE I. Power Flow Solutions

Generator (i)	E_i (p.u.)	δ_i (radians)	P_{mi} (p.u)
1	1.0566	0	0.7164
2	1.0502	0.3048	1.63
3	1.017	0.1896	0.85

Accordingly the system dynamic equation (prior to shifting the SEP to the origin) is given by the following, where δ_1 and δ_2 denote the emf angles of generators 2 and 3 respectively, relative to that of generator 1, the assumed fixed inertial frame:

$$\begin{aligned} \ddot{\delta}_1 &= 32.42 - 93.05\cos(\delta_1 - 1.373) - \\ &\qquad 81.55\cos(1.354 + \delta_2 - \delta_1) - 0.7828\dot{\delta}_1, \\ \ddot{\delta}_2 &= 32.32 - 55.75\cos(\delta_2 - 1.357) - \\ &\qquad 38.36\cos(1.354 + \delta_1 - \delta_2) - 0.1001\dot{\delta}_2. \end{aligned} \tag{32}$$

Then we perform a linear transformation to bring the SEP of (32) to the origin, followed by the non-linear transformation (14), to obtain the constrained polynomial dynamic equations with SEP at the origin, in the transformed state space of $z \in \mathbb{R}^6$.

Both the algorithms of RoA estimation are implemented in MATLAB, using SOSTOOLS 3.0 supported by SeDuMi 1.3, to solve (25)-(27), and CVX 2.1 for solving (31). For estimating the RoA using the algorithm in [11], we set the thresholds $\epsilon_\gamma = 0.001$, $\epsilon_p = 0.01$, and we kept the rest of the initializations similar to those suggested in [11], namely we set the degree of polynomials V, v_1, v_3, s_2 to 2 and that of v_2, s_1, s_3 to 0, $q_+(z) = 10^{-3} z^T z$, $r_+(z) = z^T z$, $\beta = 3$, and $p_+(z)$ is initialized to $p_+^0(z) = z^T z - 1.5 z_1 z_2 - 0.5 z_5 z_6$. For our algorithm, we kept the initializations of $p_+(z)$, $q_+(z)$, $r_+(z)$, the degree of polynomial variables, and β identical to those used in the algorithm in [11], while the values used for the other parameters are: $\alpha_{max} = 1$, $\eta = 0.5$, and $\epsilon_{\mathcal{V}} = 0.01$.

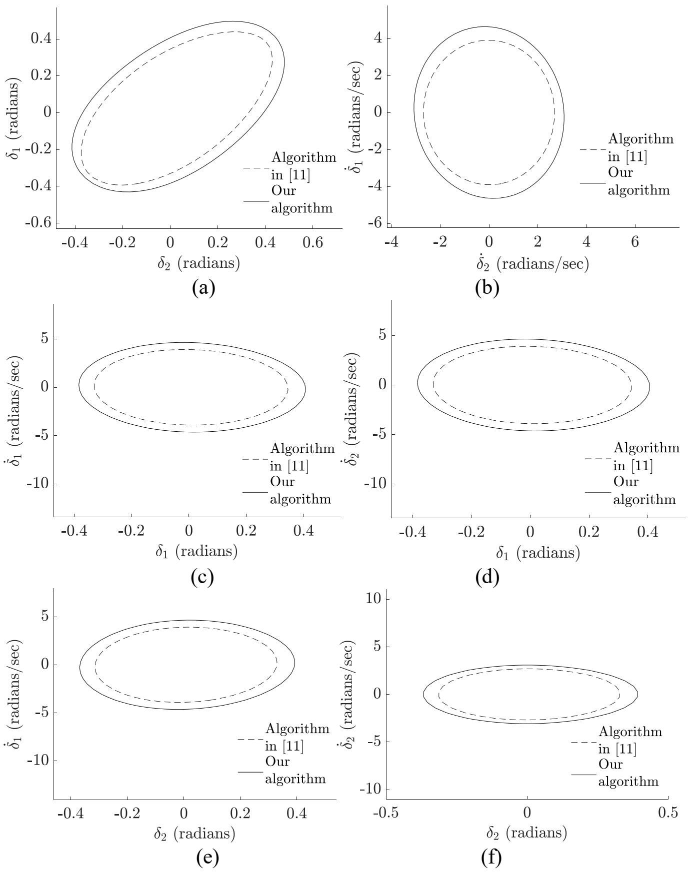

Figure 1. RoA estimated by our algorithm and that in [11]. Since the system dynamics is defined in the four dimensional space of $x = [\delta_1, \dot{\delta}_1, \delta_2, \dot{\delta}_2]$, the estimated RoA is visualized by its projections (a)-(f) on the six respective subspaces of the six possible pairs of the state variables.

After shifting the SEP to the origin by linear transformation, the RoA of (32), as estimated by the two algorithms, is shown in Fig. 1. It shows that our algorithm reports a less conservative estimate of the RoA, as compared to the algorithm in [11]. TABLE II shows the performance comparative of the two algorithms, which quantifies the enhancement offered by our algorithm, while Fig. 2 depicts their convergence curves.

TABLE II. Performance Comparative

	Algorithm in [11]	Our algorithm	%Enhanced
Computation Time	32.71 mins	10.83 mins	66.92%
Volume of RoA estimated, in the space of $z \in \mathbb{R}^6$	11.24	46.42	313%

V. Conclusion

We proposed an enhancement in the existing sum-of-squares optimization based inscribed ellipsoidal approximation to the region of attraction estimation method for power systems. We introduce a novel convex optimization based stopping criteria, which is concrete with respect to the objective of the algorithm, unlike the one in the state-of-the-art. We also suggest iterative alternation over two SoS maximizations, which results in: (i) reduction of overall computation time by approximately 66%, and (ii) estimates RoA of volume approximately 3 times to that estimated by the state-of-the-art.

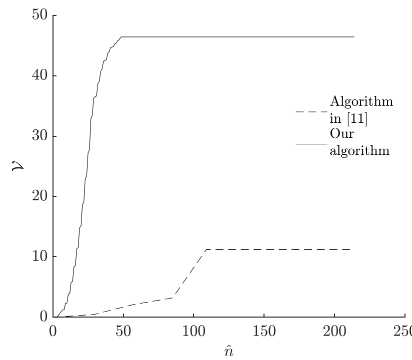

Figure 2. Iterative convergence of our algorithm and that in [11], where $\hat{n}$ denotes the total number of SoS optimizations solved by the respective algorithms (e.g. $u + v$ for our algorithm), approximately reflecting their overall computation time, and $\mathcal{V}$ denotes the estimated volume of the RoA in the space of $z \in \mathbb{R}^6$, computed using (30) setting $\mathcal{V}_u = 1$.

VI. References

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